Narges TALEBIMOTLAGH, PhD Candidate
E-mail: n.talebi@tabrizu.ac.ir
Professor Farzad HASHEMZADEH, PhD
E-mail: hashemzadeh@tabrizu.ac.ir
Professor Amir RIKHTEHGAR GHIASI, PhD
E-mail: agiasi@tabrizu.ac.ir
Professor Sehraneh GHAEMI, PhD
E-mail: ghaemi@tabrizu.ac.ir
Control Engineering Department
Faculty of Electrical and Computer Engineering:
University of Tabriz, Tabriz, Iran

A NOVEL METHOD OF MODELING DYNAMIC EVOLUTIONARY GAME WITH RATIONAL AGENTS FOR MARKET FORECASTING

Abstract. *Gold price modeling and prediction is a difficult problem and drastic changes of the price causes nonlinear dynamic that makes the price prediction one of the most challenging tasks for economists. Since gold market always has been interesting for traders, many of traders with various beliefs were highly active in gold market. The competition among two agents of traders, namely trend followers and rational agents, to gain the highest profit in gold market is formulated as a dynamic evolutionary game, where, the evolutionary equilibrium is considered to be the solution to this game. Furthermore, genetic algorithm is being used to find the unknown parameters of the model, so that we could maximize the fitness of the proposed multi agent model and the gold market daily price data. Besides the evolutionary game dynamic, we proposed a new method for modeling rational expectations using recurrent neural network. The evolutionarily stable strategies is proven despite the prediction error of the expectation. The empirical results show the high efficiency of the proposed method which could forecast future gold price precisely.*

Keywords: *Evolutionary Game Theory · Rational Agent · Evolutionary Stable State · Recurrent Neural Network · Two Step Ahead Prediction · Reinforcement Learning · Gold Market.*

JEL Classification: C70, C73, C45, C53

1. Introduction

Evolutionary game theory (EGT) reveals strategic interactions with dynamic adjustment process of players that can switch between strategies. In a typical evolutionary game, two main components are taken as payoff matrix which indicates the outcome of corresponded strategy and the dynamic rule of the agent. Over time, under dynamic rule of evolution strategies, lower associated payoffs will be replaced by strategies with higher payoffs till the strategies converge towards evolutionarily stable strategy (ESS) from the set of available strategies that is robust to evolutionary pressures and uninvadable by any other strategy [1]. Simply put, in evolutionary games, players slowly change their strategies to achieve the solution eventually [2]. When the solution to an evolutionary game has more than one equilibriums, a refined solution is required which ensures that the Nash equilibrium is stable [3] and no player can increase his payoff by choosing a different action, given other players actions [4]. In society we can conceive a selection mechanism that the strategies which perform better than average are the ones that in the long run become dominant. These dominant strategies will become the set of rules that are adopted by the majority of the population [5]. Thereupon, in the last few years , evolutionary game theory has been extensively used to model economic issues such as study the dynamics of the labor market [6],[7], study the interaction between firms and workers[8], macroeconomic monetary policy [9], neuro-economics[10] and as a tool to address the behavior in financial markets[11]. Moreover, a number of papers have investigated the stability of evolutionary dynamics and they emphasized that the large fraction of fundamentalists tends to stabilize price, whereas, a large fraction of chartists tends to destabilize price. Brock, et al. [12] investigated whether a fully rational agent can employ additional hedging instruments to stabilize markets. It turns out that the composition of the population on irrational traders and the information gathering costs for rationality may affect the answer. In this paper, the stability analyze of the dynamic is being reconsidered. We emphasize the role of heterogeneous beliefs in a market with two groups of traders having different expectations about future price. The first agent traders are fully rational with perfect foresight trying to predict the future price with neglectable error. The second typical traders are technical analysts who belief that asset prices could be predicted by simple technical trading rules, extrapolation of trends and other patterns observed in past prices. An important question in heterogeneous agents modeling is whether irrational traders can survive in the market or they would be driven out of the market by rational investors and lose their wealth. It is being proven that in a stable dynamic evolutionary game with rational agents for market forecasting, agents have nonzero fraction of the market which means that the traders would survive in the market. The major contributions of this paper can be summarized as follows:

In this paper a new approach of modeling heterogeneous evolutionary dynamic of asset pricing models with fully rational agents is proposed. Furthermore, stability condition of the proposed model has been studied. Moreover, a new method for modeling rational expectations using recurrent neural network is proposed.

Besides, the forecast ability of proposed model is become manifest modeling gold market data based on it. This model shows high convergence rate, low prediction error and efficiency in gold market forecasting. The proposed method could be widely applied to similar management and decision making problems consist of fully rational agents.

2. Evolutionary Dynamic Formulation of the Asset Pricing Model in Heterogeneous Market

The asset pricing model with heterogeneous beliefs using evolutionary selection of expectation (*BH* model) as introduced by Brock and Hommes has been used [13]. The BH model is consisting of a multi agent system where agents could either invest in a risk free asset or they could engage risky asset investment. The risk free investment pays a fixed rate of return *r*; on the contrary, the risky asset pays an uncertain dividend. Eq.1 depicts the wealth dynamic wherein $P_t$stands for the price for share of the risky asset and y_t be the stochastic dividend process of the risky asset at time t.

$$W_{t+1} = RW_t + (P_{t+1} + y_{t+1} - RP_t)Z_t \qquad (1)$$

where, $R = 1 + r$ denotes the gross rate of risk free return andZ_t denotes the number of shares of the risky asset purchased at time t. It is clear that, in a multi agent system with H different agents of traders, each agent tries to maximize the mean-variance equation with respect toZ_t to get $Z_{h,t}$ which is the number of shares purchased by agent type h.

$$\max_{Z_t} \left\{E_{h,t}[W_{t+1}] - \frac{a}{2} V_{h,t}[W_{t+1}]\right\} \qquad (2)$$

where$E_{h,t}$ and $V_{h,t}$ stand for belief or forecast of trader of agent h about conditional expectation and conditional variance respectively and a is risk-aversion parameter. Besides, Z^s,the supply of outside risky shares, assumed to be constant, $n_{h,t}$ denotes the fraction of agent type h at time$t$ and conditional variance assumed to be constant for all traders type as $V_{h,t} = \sigma^2$ the equilibrium of demand and supply yields Eq.3

$$RP_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}[P_{t+1} + y_{t+1}] - Z^s a\sigma^2 \qquad (3)$$

The term $Z^s a\sigma^2$ is the risk premium for traders to hold risky assets. Suppose, p^* denotes the common belief about the fundamental price which is equal for all traders type and x_t, deviation from the fundamental price, defined as $x_t = P_t - p^*$. In case of $E[y_t] = \bar{y}$, we assume that for all types of traders we have $E_{h,t}[y_{t+1}] = E[y_{t+1}] = \bar{y}$ and all conditional believes$E_{h,t}[P_{t+1}]$ are the form of

$$E_{h,t}[P_{t+1}] = E_{h,t}[p^*] + E_{h,t}[x_{t+1}] = p^* + f_{h,t}(x_{t-1}, x_{t-2}, \dots, x_{t-L}) \qquad (4)$$

$f_{h,t}(x_{t-1}, x_{t-2}, \dots, x_{t-L})$, the heterogeneous part of the conditional expectation, is called forecasting rule which differs agents. Now we could re-evaluate the equilibrium of supply and demand equation in deviation with respect to the $Rp^* = E_t[p^* + y_{t+1}]$ which yields

$$Rx_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}[x_{t+1}] \equiv \sum_{h=1}^{H} n_{h,t} f_{h,t} \tag{5}$$

In mentioned dynamic, the method of evolving the fraction $n_{h,t}$ of each trader type that describes how believes are updated over time is the evolutionary part of the model. Fraction is evaluated through the multi-nominal logit model called Gibbs probabilities which is based on the discrete choice models.

$$n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}} \tag{6}$$

where, β shows the intensity of choice depends on the sensitivity of traders to selecting the optimal prediction strategy and $U_{h,t}$ is the realized profit of traders type h which is a natural candidate for evolutionary fitness. If $,0 \leq \eta \leq 1$, is a memory parameter in fitness function showing the impact of past realized fitness on strategy selection. In case of zero supply of outside risky shares,$Z^s = 0$, fitness function can be rewritten as

$$U_{h,t} = (x_t - Rx_{t-1})\left(\frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}\right) + \eta . U_{h,t-1} \tag{7}$$

2.1 Evolutionary Model with Fully Rational Agents

In this part, an evolutionary dynamic with a rational agent will be investigated. In first agent, traders try to have a perfect prediction about future price which leads to development of fully rational agent with perfect foresight. They have perfect knowledge of heterogeneous market equilibrium equation and beliefs of all other traders. This agent's forecasting rule is obtained by Eq.8

$$f_{1,t} = x_{t+1} \tag{8}$$

In contract to rational traders, trend followers believe that price varies in a very simple manner with respect to previous data. They use linear forecasting rule that is given by Eq.9

$$f_{2,t} = g . x_{t-1} \tag{9}$$

Eq.11 will be formed with substituting the beliefs of the agents with perfect foresight and trend followers as in Eq.8and Eq.9 into Eq.5.

$$R.x_t = n_{1,t-1}.x_{t+1} + n_{2,t-1}.g.x_{t-1} \tag{10}$$

where,$n_{h,t-1}$ is the fraction of agent type h at time $t-1$.

Suppose, m is defined as the difference of n_1 and $n_2$as $m = n_1 - n_2$. Knowing $n_1 + n_2 = 1$ and $n_{1,t} = \frac{1+m_t}{2}$ and $n_{2,t} = \frac{1-m_t}{2}$, Eq.11 could be rewritten as

$$Rx_t = \frac{1+m_{t-1}}{2}.x_{t+1} + \frac{1-m_{t-1}}{2}.g.x_{t-1} \tag{11}$$

Carrying x_{t+1} to the other side of the equation, Eq.12 leads to Eq.13

$$x_t = \frac{2R}{1+m_{t-2}}.x_{t-1} + \frac{m_{t-2}-1}{1+m_{t-2}}.g.x_{t-2} \tag{12}$$

Evaluating fitness function for both agents through Eq.7 and substituting them in Eq.6, the dynamic of m could be written as

$$m = n_1 - n_2 = tanh\left[\frac{\beta}{2}\left\{\frac{1}{a\sigma^2}\left[\left(\frac{2R}{1+m_{t-2}} - R\right).x_{t-1} + \frac{m_{t-2}-1}{1+m_{t-2}}.g.x_{t-2}\right].\left[\frac{2R}{1+m_{t-2}}.x_{t-1} + \left(\frac{m_{t-2}-1}{1+m_{t-2}} - 1\right).g.x_{t-2}\right] - C\right\}\right] \tag{13}$$

Note that, Eq.12 and Eq.13 represent a nonlinear system dynamic and if the states of the mentioned dynamics are being chosen as in Eq.14, the global dynamic of the system could be easily analyzed.

$$X(k) = [X_1(k), X_1(k), X_1(k), X_1(k)] = [m_{t-2}, m_{t-1}, x_{t-2}, x_{t-1}] \tag{14}$$

Accordingly, the nonlinear state space equations could be written as

$$X_1(k+1) = X_2(k) \tag{15.a}$$

$$X_2(k+1) = tanh\left[\frac{\beta}{2}\left\{\frac{1}{a\sigma^2}\left[\left(\frac{2R}{1+X_1(k)} - R\right).X_4(k) + \frac{X_1(k)-1}{1+X_1(k)}.g.X_3(k)\right].\left[\frac{2R}{1+X_1(k)}.X_4(k) + \left(\frac{X_1(k)-1}{1+X_1(k)} - 1\right).g.X_3(k)\right] - C\right\}\right] \tag{15.b}$$

$$X_3(k+1) = X_4(k) \tag{15.c}$$

$$X_4(k+1) = \frac{2R}{1+X_1(k)}.X_4(k) + \frac{X_1(k)-1}{1+X_1(k)}.g.X_3(k) \tag{15.d}$$

2.1.1 Stability Analysis

Definition 1: An equilibrium point E is evolutionary stable state(ESS) of a system if the system in that state cannot be invaded by any new mutant strategies [14].

Theorem 1: If E is an ESS then it is strictly stable equilibrium point of the discrete dynamical system restricted to k. note that, in a sampled data system, a fixed point is stable if all eigenvalues have a negative real part [1].

Here, if we consider the evolutionary equilibrium as the solution to this dynamic, the stability of evolutionary equilibrium should be analyzed. The fixed point of the dynamic is the evolutionary equilibrium where it is obtained by solving $\bar{X}(k+1) = \bar{X}(k)$.

For evaluating the fixed points of the system, the equality $\bar{X}(k+1) = \bar{X}(k)$ ought to be solved, where, $\bar{X} = [X_1; X_2; X_3; X_4]$and leads to

$$\bar{X}_1 = \bar{X}_2 \tag{16.a}$$

$$\bar{X}_2 = tanh\left[\frac{\beta}{2}\left\{\frac{1}{a\sigma^2}\left[\left(\frac{2R}{1+\bar{X}_1} - R\right).\bar{X}_4 + \frac{\bar{X}_1-1}{1+\bar{X}_1}.g.\bar{X}_3\right].\left[\frac{2R}{1+\bar{X}_1}.\bar{X}_4 + \left(\frac{\bar{X}_1-1}{1+\bar{X}_1} - 1\right).g.\bar{X}_3\right] - C\right\}\right] \tag{16.b}$$

$$\bar{X}_3 = \bar{X}_4 \tag{16.c}$$

$$\bar{X}_4 = \frac{2R}{1+\bar{X}_1}.\bar{X}_4 + \frac{\bar{X}_1-1}{1+\bar{X}_1}.g.\bar{X}_3 \tag{16.d}$$

Now, with consideration of $\bar{X}_1 = \bar{X}_2 = \bar{m}$and $\bar{X}_3 = \bar{X}_4 = \bar{x}$, we solve Eq.16

$$\bar{m} = tanh\left[\frac{\beta}{2}\left\{\frac{1}{a\sigma^2}\left[\left(\frac{2R}{1+\bar{m}} - R\right).\bar{x} + \frac{\bar{m}-1}{1+\bar{m}}.g.\bar{x}\right].\left[\frac{2R}{1+\bar{m}}.\bar{x} + \left(\frac{\bar{m}-1}{1+\bar{m}} - 1\right).g.\bar{x}\right] - C\right\}\right] \tag{17.a}$$

$$\bar{x} = \frac{2R}{1+\bar{m}}.\bar{x} + \frac{\bar{m}-1}{1+\bar{m}}.g.\bar{x} \tag{17.b}$$

Eq. 17.b leads to$\bar{x} = 0$or $\bar{m} = 1 - 2\frac{R-1}{g-1}$. Now with substituting these solutions in Eq. 17.a the equilibrium points will be found. Substituting $x^{eq} = 0$ in Eq.17.a and solving the equation with respect to m, yields into $m^{eq} = tanh({}^{-\beta C}/_{2})$, which is one of the equilibrium poin*ts of the dynamic that is* $E_1 = [m^{eq}, m^{eq}, x^{eq}, x^{eq}]$ =[$tanh({}^{-\beta C}/_{2})$, $tanh({}^{-\beta C}/_{2})$, 0,0]. Now, with substitution of $m^* = 1 - 2\frac{R-1}{g-1}$ in Eq. 16.a and solving the equation with respect to x, the answer x^*, would be the solution of the $1 - 2\frac{R-1}{g-1} = tanh\left\{\frac{\beta}{2}[D(g-1)(R-1)(x^*)^2 - C]\right\}$.

If x^* would be the positive solution of the mentioned equality and based on the fact that this equation has two conjugate roots, then there could exist two fixed point of the dynamic which are $E_2 = [m^*, m^*, x^*, x^*]$ and $E_3 = [m^*, m^*, -x^*, -x^*]$

where, $x^* = \frac{\sqrt{2C+ln(\frac{g-R}{R-1})}}{\sqrt{\beta D(R-1)(g-1)}}$. As considered before, $-1 \leq m \leq 1$. According to $\overline{m} = 1 - 2\frac{R-1}{g-1}$, $m > 1$ and $m < -1$ lead to $g < R$ in which Eq.17 doesn't have any solution which means if $g < R$, E_1 is the unique equilibrium points of the dynamic. For $g > 2R - 1$ there exists three equilibrium points $E_1, E_2$and E_3. Besides, for $R < g < 2R - 1$ there are two possibilities. If $(1 - 2\frac{R-1}{g-1}) < tanh\left({-\beta C}/{2}\right), E_1$ is the unique equilibrium point. Else, there exists three equilibrium points $E_1, E_2$and E_3.

For the system of the form:

$$x(k+1) = f(x(k)) \tag{18}$$

Recall that, if a function $\psi: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing and $\psi(0) = 0$ then it is a $\mathcal{K} - function$. Furthermore, it is a $\mathcal{K}_\infty - function$ if it is a $\mathcal{K} - function$ and also $\psi(s) \to \infty$as $s \to \infty$. If $\psi(s) > 0$ for all $s > 0$, and $\psi(0) = 0$, it is a positive definite function. A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL} - function$ if for each fixed $s \geq 0$, the function $\beta(s,.)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$, and for each fixed $t \geq 0$, the function $\beta(.,t)$ is a $\mathcal{K} - function$.

Note that in a stable system, every state trajectory remains bounded; and no matter what the initial state is, the state trajectory eventually becomes small.

Definition 2: A nonlinear dynamic system of Eq.18 is asymptotically stable (AS) if there exist a $\mathcal{KL} - function$, $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for each $\xi \in \mathbb{R}^n$, it holds $|x(k,\xi)| \leq \beta(|\xi|, k)$ for each $k \in \mathbb{Z}_+$.

Definition 3: A continuous function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a *lyapunov function* for a nonlinear dynamic system if the following holds:

There exist $\mathcal{K}_\infty - function \alpha_1$ and α_2 such that

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|), \quad \forall \xi \in \mathbb{R}^n \tag{19}$$

There exist a a$\mathcal{K}_\infty - function \alpha_3$ such that

$$V(f(\xi)) - V(\xi) \leq \alpha_3(|\xi|) \forall \xi \in \mathbb{R}^n \tag{20}$$

Theorem 2: The linear discrete-time system is considered as

$$x(k+1) = \mathrm{A}x(k) \tag{21}$$

where the eigenvalues of A are located strictly inside the unit disk. For a symmetric and positive-definite matrix Q, $P > 0$ is the unique solution to the matrix $A^TPA - A = -Q$. The matrix $V(x) = x^TPx$ is positive-definite and radially unbounded function which satisfies the condition of Definition 3 with $\alpha_1(r) = \lambda_{min}(P)r^2$, $\alpha_2(r) = \lambda_{max}(P)r^2$ and $\alpha_3(r) = \frac{1}{2}\lambda_{min}(Q)r^2$. Therefore, V is a $Lyapunov\ function$ for the system in Eq.21 [15].

Corollary: system in Eq.18 is asymptotically stable in fixed point E if the eigenvalues of the Jacobian matrix stays inside unit disk.

It has been proven that, an evolutionary dynamic converges to fundamental equilibrium which is why the stability of E_1 will be discussed. The Jacobian matrix of the nonlinear dynamic in Eq.15 is formed at the fixed point E_1.

$$J = \begin{bmatrix} \frac{\partial X_1[k+1]}{\partial X_1[k]} & \cdots & \frac{\partial X_1[k+1]}{\partial X_4[k]} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_4[k+1]}{\partial X_1[k]} & \cdots & \frac{\partial X_4[k+1]}{\partial X_4[k]} \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & 0 & 1 \\ 0 & 0 & \frac{g(1+tanh(^{\beta C}/_2))}{tanh(^{\beta C}/_2)-1} & \frac{2R}{tanh(^{\beta C}/_2)-1} \\ 0 & 0 & & \end{pmatrix} \tag{22}$$

The eigenvalues of the mentioned matrix are found to be

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{-R+\sqrt{R^2-g+g.tanh^2(\frac{C\beta}{2})}}{tanh\left(\frac{C\beta}{2}\right)-1} \\ \frac{-R-\sqrt{R^2-g+g.tanh^2(\frac{C\beta}{2})}}{tanh\left(\frac{C\beta}{2}\right)-1} \end{pmatrix} \tag{23}$$

where, $1 \le R \le 2, \beta \ge 0$, $C \ge 0$ and $0 \le [A = tanh\left(\frac{C\beta}{2}\right)] \le 1$. To establish stable equilibrium, g have to be evaluated in such way that eigenvalues stay inside unit circle. As it is shown in Eq.23, the J matrix, has four eigenvalues which contains two zero eigenvalues that are permanently inside unit disk. The other two should be revisited.

2.2 Evolutionary Model With Partly Rational Agents

A heterogeneous evolutionary model of pricing including an agent with perfect foresight has been investigated earlier. The problem with perfect foresight is that many researchers think the perfect forecast assumption is unrealistic. Possessing rational forecast under homogeneous expectations would require knowledge of the law of motion. But it is even more demanding in the heterogeneous world, where one should also know what others expect. In other words, a perfect forecaster would has to know the whole dynamic of the system and the expectations of other agents about future price to make a precise two step ahead (2SA) predict of future price. Besides, a mistaken 2SA predict could affect the stability of equilibrium points of the model. Here, it is being assumed that the future price has been estimated with a reliable method and we discuss the robustness of the model with respect to estimation error. Suppose two step ahead price has been forecasted with error ε which means Eq.8 changes into Eq.24

$$f_{1,t} = x_{t+1} + \boldsymbol{\varepsilon_t} \tag{24}$$

This error parameter causes some variations in state space equations as in Eq.25

$$X_1[k+1] = X_2[k] \tag{25.a}$$

$$X_2[k+1] = tanh\left[\frac{\beta}{2}\left\{\frac{1}{a\sigma^2}\left[\left(\frac{2R}{1+X_1[k]} - R\right).X_4[k] + \frac{X_1[k]-1}{1+X_1[k]}.g.X_3[k] + \boldsymbol{\varepsilon}[\boldsymbol{k}]\right].\left[\frac{2R}{1+X_1[k]}.X_4[k] + \left(\frac{X_1[k]-1}{1+X_1[k]} - 1\right).g.X_3[k] + \boldsymbol{\varepsilon}[\boldsymbol{k}]\right] - C\right\}\right] \tag{25.b}$$

$$X_3[k+1] = X_4[k] \tag{25.c}$$

$$X_4[k+1] = \frac{2R}{1+X_1[k]}.X_4[k] + \frac{X_1[k]-1}{1+X_1[k]}.g.X_3[k] + \boldsymbol{\varepsilon}[\boldsymbol{k}] \tag{25.d}$$

Suppose $\boldsymbol{\varepsilon}[\boldsymbol{k}]$ is small enough, such that the fixed points of the system stay unchanged. For the discrete time nonlinear system of the form

$$X[k+1] = f(X[k], d[k]) \tag{26}$$

where, $d(k)$is the disturbance or time varying parameter.

Corollary: It is proven that a discrete-time system with disturbances or time-varying parameters, taking values in a compact set, is uniformly asymptotically stable (*UAS*) with respect to a closed, not necessarily compact, invariant set $\mathcal{A}$ if and only if there exists a smooth *Lyapunov function* V with respect to the set $\mathcal{A}$ [16]. The system presented in Eq.25 could be presented as a form of Eq.26 assuming $d(k) = \boldsymbol{\varepsilon}[\boldsymbol{k}]$. The mentioned system is stable in fixed point E if the eigenvalues of the Jacobian matrix stays inside unit disk based on Theorem 2 and the closed and invariant set $\mathcal{A}$ is the Region of Attraction (*ROA*) of the system.

Here, we compute the Jacobian matrix of the new dynamic in E_1 and calculate the eigenvalues of it. Here, if we try to analyze the stability of the model with consideration of $\boldsymbol{\varepsilon}$**,** eigenvalues of the Jacobian matrix will be found.

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} = \left(\begin{array}{c} 0 \\ 0 \\ \dfrac{R-R.tanh\left(\frac{C\beta}{2}\right)-tanh\left(\frac{\beta(C-D\varepsilon^2)}{2}\right).A+A+R.tanh\left(\frac{\beta D\varepsilon^2}{2}\right)-R.tanh\left(\frac{\beta D\varepsilon^2}{2}\right).tanh\left(\frac{C\beta}{2}\right)}{\left(tanh\left(\frac{\beta D\varepsilon^2}{2}\right)+1\right).\left(tanh\left(\frac{\beta(C-D\varepsilon^2)}{2}\right)-1\right).\left(tanh\left(\frac{C\beta}{2}\right)-1\right)} \\ \dfrac{R-R.tanh\left(\frac{C\beta}{2}\right)+tanh\left(\frac{\beta(C-D\varepsilon^2)}{2}\right).A-A+R.tanh\left(\frac{\beta D\varepsilon^2}{2}\right)-R.tanh\left(\frac{\beta D\varepsilon^2}{2}\right).tanh\left(\frac{C\beta}{2}\right)}{\left(tanh\left(\frac{\beta D\varepsilon^2}{2}\right)+1\right).\left(tanh\left(\frac{\beta(C-D\varepsilon^2)}{2}\right)-1\right).\left(tanh\left(\frac{C\beta}{2}\right)-1\right)} \end{array}\right), \tag{27}$$

and $, A =$

$$\sqrt{\begin{array}{l} R^2.tanh^2\left(\frac{C\beta}{2}\right).tanh^2\left(\frac{\beta D\varepsilon^2}{2}\right)-2R^2.tanh\left(\frac{\beta D\varepsilon^2}{2}\right).tanh\left(\frac{C\beta}{2}\right)+R^2 \\ -g.tanh^2\left(\frac{\beta D\varepsilon^2}{2}\right).tanh^2\left(\frac{C\beta}{2}\right)+g.tanh^2\left(\frac{\beta D\varepsilon^2}{2}\right)+g.tanh^2\left(\frac{C\beta}{2}\right)-g \end{array}}$$

As it is shown in Eq.27, the J matrix, has four eigenvalues which contains two zero eigenvalues that are permanently inside unit disk. The other two should be revisited.

3. A Novel Method in Two-Step-Ahead Market Forecasting

In this section, for *2SA* predicting of price an Improved Reinforced Real Time Recurrent Learning (*IR-RTRL*) algorithm for Recurrent Neural Networks (*RNNs*) is presented. Reinforced Recurrent Neural Network (*R-RNN*) as which is depicted in figure 1, consists of 2 layers, M external inputs and K outputs [17].

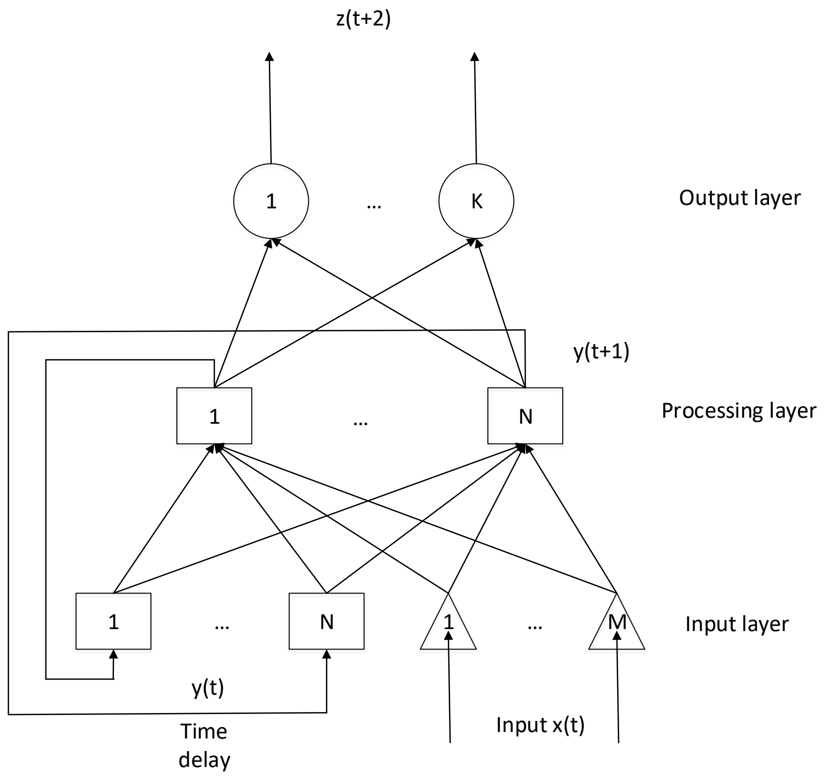


Figure 1. Architecture of 2SA RNN

The network input, as illustrated in Eq.40 is formed of two vectors $x(t)$and $y(t)$, where $x(t)$ denotes the $M \times 1$ discrete time varying input vector and $y(t+1)$ is the $N \times 1$ output of the corresponding processing layer.

$$\mu(t) = [\, x(t)\, ;\, y(t)\,] \tag{28}$$

On the other hand, as shown in figure 1, $y(t+1)$ is the input of the second layer and $z(t+2)$ denotes the corresponding $k \times 1$ output.The output of neuron j in the processing layer and the net output of neuron k in the output layer at time $t+2$are given by Eq.29.a and Eq.29.b respectively.

$$y_j(t+1) = f\left(\textstyle\sum_{i \in A \cup B} w_{1ji}(t)\mu_i(t)\right) \quad (29.a)$$
$$z_k(t+2) = f\left(\textstyle\sum_j w_{2kj}(t+1)y_j(t+1)\right) \quad (29.a)$$

where, $w_{lji}(t)$ is network weight and $f(.)$ is a nonlinear function. In output layer, the instantaneousoverall network error is defined in Eq.30, where $e_k(t+2)$ is the kth element of time-varying $K \times 1$ error vector and $d_k(t+2)$ denotes the target value of neuron k at time $t+2$.

$$E(t+2) = \frac{1}{2}\sum_{k=1}^{K} e_k^2(t+2) = \frac{1}{2}\sum_{k=1}^{K}(d_k(t+2) - z_k(t+2))^2 \tag{30}$$

The weight change for any particular weight $\mathrm{w_{lmn}}$ update rule, yields based on gradient descent back propagation method, can be written as Eq.31.

$$\Delta w_{lmn}(t) = -\eta_{adaptive} \frac{\partial_{(\alpha+\beta.DS)}\big(E(t+2)\big)}{\partial w_{lmn}(t)} \tag{31}$$

whereα and β are constant coefficient and DS is directional parameter. Even though minimizing the prediction error and making an accurate forecast is very important, predicting the direction of movement of financial time series has higher importance. Moreover, as discussed before, customers have to arrange their decision of trading which affect their benefits and total wealth. Furthermore, correct forecasting directions or turning points between the actual and predicted values could lead them toward improved decisions of trading. Thus, based on the fact that direction prediction plays an essential role in efficiency of market forecasting methods, an improved method of learning is presented, in which, an improved punishment function is proposed which include a linear coefficient depending on the DS which is tend to be used as an evaluation criterion of direction prediction thus far [18],[19]. $DS$is a statistical measure of a model's performance in predicting the direction of change, positive or negative, of a time seriesfrom one time period to the next.

$$DS = \frac{1}{M}\sum_{t=1}^{N} a(t) \times 100\% \tag{32}$$

where, M is the length of input signal and parameter $a(t)$ is defined through Eq.33

$$a(t) = \begin{cases} 1 & if \ \big(d(t+3) - d(t+2)\big)\big(z(t+3) - z(t+2)\big) > 0 \\ 0 & otherwise \end{cases} \tag{33}$$

Improved reinforced *2SA* weight adjustment procedure for *RNN* is represented completely in figure 2.

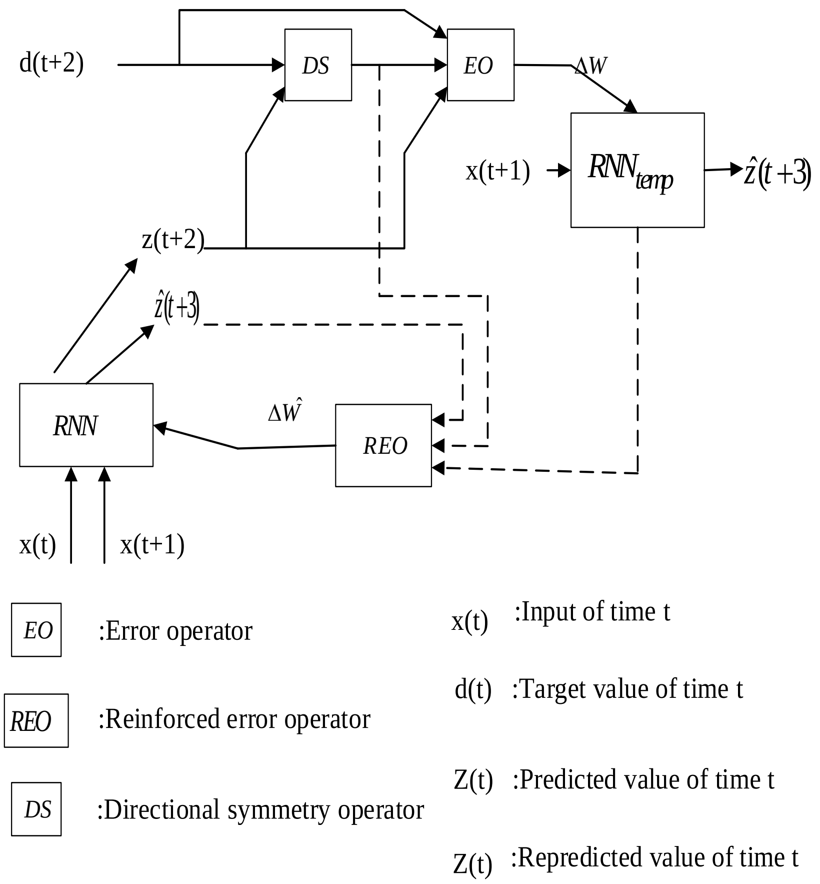


Figure 2. Improved reinforced 2SA weight adjustment procedure for RNN

Following that, for the reinforcement learning stage we have the set of Eq.34

$$\hat{y}_j(t+2) = f\left(\sum_{i \in A \cup B} \left(w_{1ji}(t) + \Delta w_{1ji}(t)\right) \mu_i(t+1)\right)$$

$$\hat{z}_k(t+3) = f\left(\sum_{j} \left(w_{2ji}(t+1) + \Delta w_{2ji}(t+1)\right) \hat{y}_j(t+2)\right)$$

$$\hat{e}_k(t+3) = \hat{z}_k(t+3) - z_k(t+3)$$

$$\hat{E}(t+3) = \frac{1}{2}\sum_{k=1}^{K} \hat{e}_k^2(t+3)$$

$$\Delta\hat{w}_{lmn} = -\eta_{adaptive}\frac{\partial_{(\alpha+\beta.dsf)}\left(\hat{E}(t+2)\right)}{\partial w_{lmn}} \tag{34}$$

And finally, the updating process is given by Eq.35

$$w_{lmn}^{new} = w_{lmn} + \Delta w_{lmn} + \Delta\hat{w}_{lmn} \tag{35}$$

The training phase algorithm for the *ImprovedR-RTRL* (*IR-RTRL*) could be wrapped up as below:

i. Initialize the network(RNN)
ii. Apply input $x(t)$ to the RNN and get corresponding output $z(t+2)$
iii. Compare $z(t+2)$ with desired output $d(t+2)$ and get $E(t+2)$
iv. Calculate DS
v. Update weights based on gradient method over punishment function $(\alpha+\beta.DS)\big(E(t+2)\big)$ with the adaptive learning rate and get the temporal neural network, RNN_{temp}
vi. Apply input $x(t+1)$ to RNN_{temp} and get corresponding output $\hat{z}(t+3)$
vii. Compare $\hat{z}(t+3)$ with desired output $d(t+3)$ and get $\hat{E}(t+3)$
viii. Update weights based on gradient method over punishment function $(\alpha+\beta.dsf)\left(\hat{E}(t+2)\right)$ with the adaptive learning rate and get RNN for next iteration
ix. Go to step 2

Figure 3 depicts flow chart of the online learning scheme for the *IR-RTRL*. As shown in the figure, after initializing network and parameters, a specified number of past data forms a time window of data which is used as network input. After training network for a specified input, to modify the structure of the training algorithm, the correctness of direction is being checked and in case of wrong detection, the training stage reiterates. And the algorithm continues with the input data of next iteration.

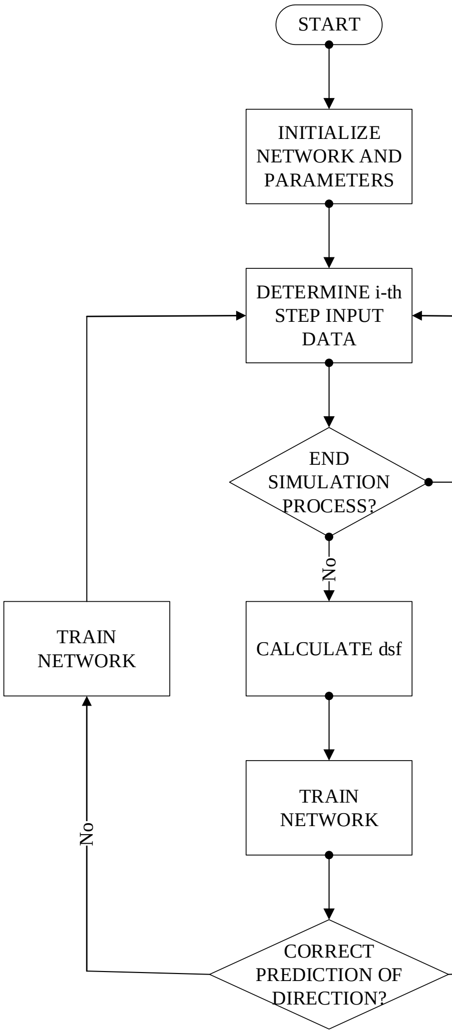


Figure 3. Flow chart of the online learning algorithm for I-RRNN

4. Application

We apply the proposed model to daily gold price data from the database of Bloomberg, which is the open access database including historical data of gold market is used to perform simulations. In this work, spanning data from 21 December 2012 to 15 August 2014, a total of 1131 observations are used in genetic algorithm method to find the unknown parameters of model and spanning data from 16 August 2014 to 15 January 2016, a total of 427 observations are used for testing the model. Besides, spanning data from 21 December 2012 to 15 January

2016, a total of 1558 observations are used in online training of the neural network. In learning stage of *RNN*, more importance is given to the new data by weakening the influence of older data points. In each level of online learning, the modification of weights is performed on a time window which contains the specified number of data till considered point. The process in Eq.36 helps us to normalize the data series.

$$X_{normalized} = 0.1\, log\big(X(t)\big) \quad (36)$$

We depict the predominance of the proposed neural network by comparing the performance of it with *R-RTRL* network and *BPNN*. The forecast ability of the model is become manifest by studying the Mean Square Error (*MSE*) and the Normalized Mean Square Error (*NMSE*) criterions given by Eq.37 and Eq.38.

$$MSE = \frac{1}{M}(d(t) - o(t))^2 \quad (37)$$

$$NMSE = \frac{(d(t)-o(t))^2}{(o(t)-\bar{o}(t))^2} \quad (38)$$

where, M is the length of input signal and $\bar{o}$ is the average of the observed values and $d(t)$ and $o(t)$ denote the predicted and observed value of price at time t respectively.

To model behavior of an economical system based on its time series data set, the set of unknown parameters $\beta, R\,, g\,, D\,, C$ and p^* should be evaluated. For a sample system like gold price, considering Eq.23, the condition in Eq.39 should be satisfied.

$$MinP_T = \frac{1}{2}\sum_{t=1}^{T}(d(t) - o(t))^2$$
$$Subjectto: \; |\lambda_3| < 1 \& |\lambda_4| < 1 \quad (39)$$

where, $d(t)$ and $o(t)$denote the predicted and observed value of price at time t respectively and P_T is the estimation error over time. The parameters β,R ,g ,D ,C andp^* are calculated using the standard genetic algorithm applied over time series of gold market so that the condition in Eq.39 would be satisfied. The parameters will be found as $\beta = 0.27$, $R = 1.02$, $g = 1.047$, $D = 1717.8$, $C = -0.18$ and $p^* = -0.73$. Assuming proposed parameters, the stability condition of three fixed points of the dynamic has been studied. For E_1, the characteristic polynomial of the Jacobian matrix would be $d(\lambda) = \lambda^4 - 1.9938\lambda^3 + 0.9972\lambda^2$ which means the eigenvalues of the matrix are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0.9969 - 0.0581i$ and $\lambda_4 = 0.9969 + 0.0581i$. It is clear that all eigenvalues stay inside unit circle that proves the stability of the system in E_1. For $E_2$and E_3, the solutions of the characteristic polynomial would be $\lambda_1 = 1.067 + i0.027$, $\lambda_2 = 1.067 - i0.027$, $\lambda_3 = 0.14 + i\,0.22$ and $\lambda_4 = 0.14 - i\,0.22$, which means two of four eigenvalues of the matrix are not inside unit circle that proves the instability of the system in E_2 and E_3. Thus, $E_1$is the only stable equilibrium and a regular ESS of the system.

4.1Evolutionary equilibrium with respect to varying parameters

In this section, the stability of E_1 will be discussed according to varying parameters. As discussed in previous section, two of eigenvalues are always constant and equal to zero. That is why considering the other two eigenvalues is adequate. As a first step, parameter β changes considering g, D, C andp^* constant.

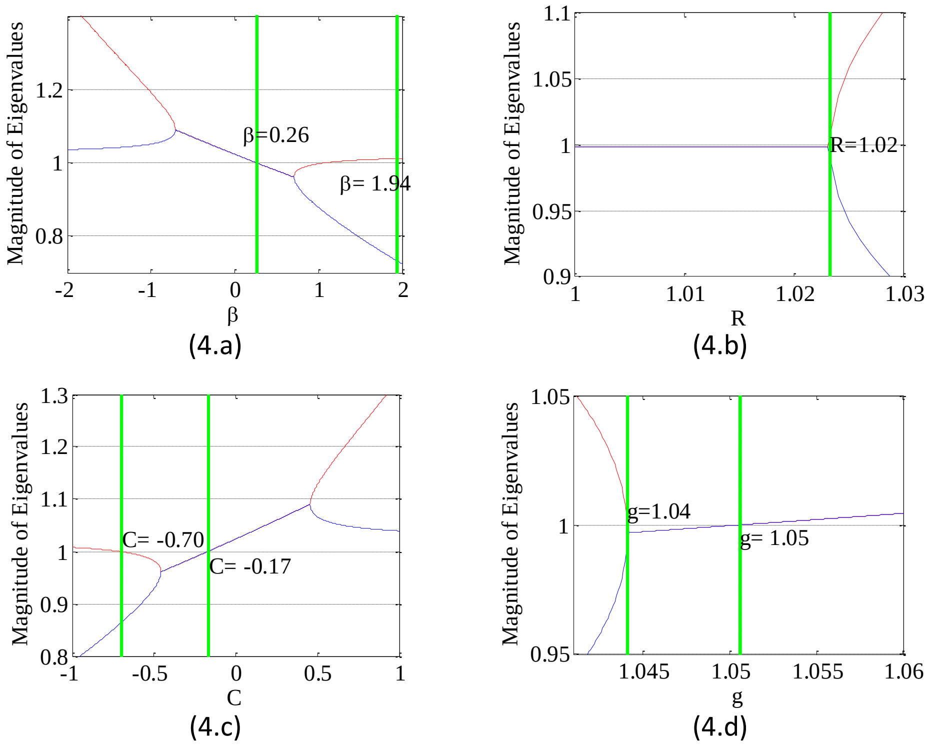


Figure 4: Magnitude of eigenvalues with respect to varying parameters. Numerical analysis: Case $\beta = 0.27$, $R = 1.02$, $g = 1.047$, $D = 1717.8$, $C = -0.18$ and $p^* = -0.73$

As depicted in figure (4.a), if β remains in [0.26 1.94] area, system remains stable. It means for large values of β, i.e. high intensity of choice, system gets unstable. Figure (4.b) shows that for small amounts of risk free return, $1 \leq R \leq 1.02$ the system stays stable. But as the amounts of risk free return grows, investors would rather to invest on risk free asset which makes this system unstable. Figure

(4.c) demonstrates the variation of eigenvalues with respect to varying C. This parameter shows the benefit that rational agent gets. It is clear that rational agent should spend profit to get information for being rational. The stable area for parameter $C$is $-0.71 \leq C \leq -0.17$. According to figure (4.d), if the belief of the technical traders stays which affects parameter g stays in the range [1.04 1.05], system remains stable.

According to Eq.27, it is quite clear that the Jacobian matrix of an evolutionary model with partly rational agents has only two nonzero eigenvalues which are continuous with respect to term ε. Analyzing the eigenvalues with respect to varying parameter ε, one could conclude that these two eigenvalues will stay inside unit circle for small ε. That means one could claim that if any agent's belief of future price has prediction error less than ε_0 the equilibrium remains stable; which is depicted in figure 5 to be $|\varepsilon_0| = 0.02$.

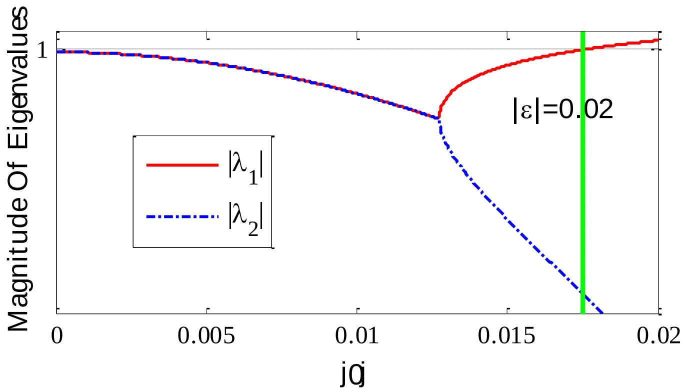
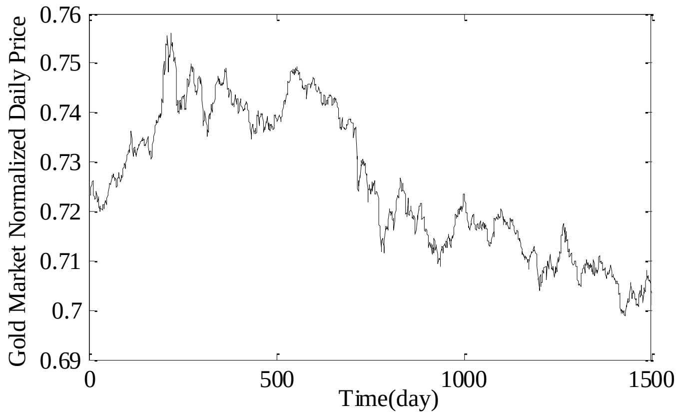


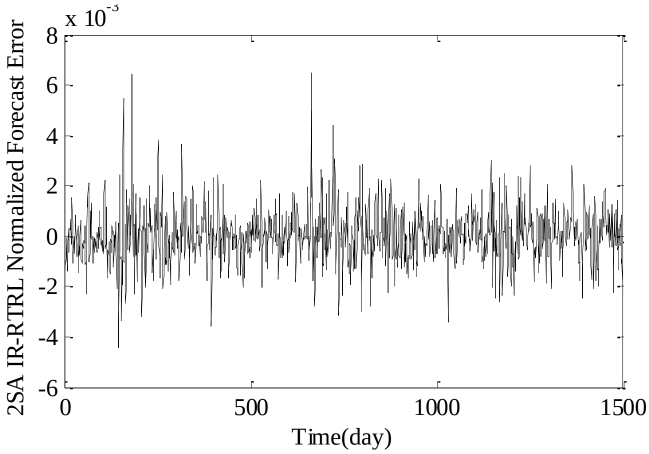
Figure 5. Magnitude of the eigenvalues of system with partially rational agent

Results demonstrated in table 1, depicts that the proposed*IR-RTRL* algorithm has smaller *NMSE* and *MSE*in *2SA* forecast. According to the table 1, *MSE* of online *R-RTRL* learning is larger than twice the *MSE* of proposed online *IR-RTRL* learning method.

Figure 6 depicts the gold market price and online *2SA*normalized forecast error of it based on proposed *IR-RTRL*. As clarified in figure 6, absolute value of the forecasting error of the proposed method does not exceed the stability margin $\boldsymbol{\varepsilon_0}$,of the model and one could draw the conclusion that in case of applying it as *2SA* forecasting method, the equilibrium remains stable and this prediction could be used as an estimation of *2SA* price expectation in heterogeneous agent models for gold market modeling.



(a)



(b)

Figure 6. (a) Gold market normalized daily price and (b) online 2SA normalized forecast error of it based on Proposed IR-RTRL

Table 1 Model performance of 1SA and 2SA forecast of gold market price

	1SA forecast		*2SA* forecast	
	MSE	*NMSE*	*MSE*	*NMSE*

EGT	1.84E+02	3.82E-03	---	---
IR-RTRL	2.12E+02	4.39E-03	2.17E+02	4.50E-03
R-RTRL	3.57E+02	7.43E-03	3.64E+02	7.56E-03
BPNN	6.73E+02	1.42E-02	6.80E+02	1.43E-02

Figure 7 depicts the online forecast of Gold marketbased on the proposed evolutionary model along with Gold market daily price. Clearly, the forecast error of the evolutionary model is significantly small. Moreover, studying the results displayed in table 1 shows that 1SA forecast based on the evolutionary model has lower mean square error and normalized mean square error comparing to other methods and in conclusion, the proposed model is efficient. As a conclusion, the evolutionary model performs an efficient and precise prediction of future gold price forecasting issue.

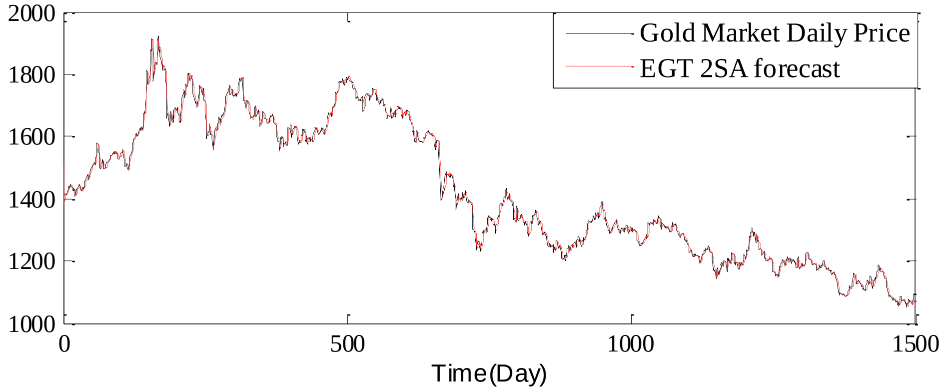


Figure 7. Gold market daily price along with EGT 2SA

5. Conclusion

In this paper, a novel method of modeling an evolutionary dynamic consisting of rational agent has been proposed and approved stable. This novel approach has been proposed to solve the problem with modeling future dependency of rational agent consistent dynamics. Furthermore, the stability of the dynamic with partly rational agent which has imprecise prediction has been analyzed and the precise analysis of stability of the equilibrium lead into finding the maximum value for prediction error that does not affect the stability has been evaluated. Furthermore, proposed approach could be used when an agent has a complicated belief that may cause complexity in dynamic and stability analyze of the model.

REFERENCES

[1] **Taylor, Peter D., and Leo B. Jonker (1978),** ***Evolutionary Stable Strategies and Game Dynamics.*** *Mathematical biosciences*, *40*(1), 145-156;

[2] **Fisher, R. A. (1930),** ***The Genetic Theory of Natural Selection;*** U.K.:Clarendon: Oxford;

[3] **Sahlins, M. D. (1970),** ***Evolution and Culture.*** Ann Arbor, MI: Univ. Michigan;

[4] **Tadj, L. a. (Nov. 2003),** ***A QBD Approach to Evolutionary Game Theory*****.** *Applied Mathematical Modelling*, *27*(11), 913-927;

[5] **Villena, M. G. (2004),** ***Evolutionary Game Theory and Thorstein Veblen's Evolutionary Economics: Is EGT Veblenian?*** *Journal of Economic Issues*, 585-610;

[6] **Araujo, Ricardo Azevedo, and Helmar Nunes Moreira (2014),*****Lyapunov Stability in an Evolutionary Game Theory Model of the Labour Market.****EconomiA*, *15*(1), 41-53;

[7] **Araujo, Ricardo Azevedo, and Nathalia Almeida de Souza (2010),** ***An Evolutionary Game Theory Approach to the Dynamics of the Labour Market: A Formal and Informal Perspective.*** *Structural Change and Economic Dynamics*, *21*(2), 101-110;

[8] **Gamba, Elvio Accinelli, and Edgar J. Sánchez Carrera. (2010),*****The Evolutionary Processes for the Populations of Firms and Workers.****Ensayos Revista de Economia*, *29*(1), 39-68;

[9**] d'Artigues, Agnés and Thierry Vignolo (2005),** ***An Evolutionary Theory of the Convergence towards Low Inflation Rates.*** *Journal of Evolutionary Economics*, *15*(1), 51-64;

[10] **Schipper, B. C. (2008),** ***On an Evolutionary Foundation of Neuroeconomics.****Economics and Philosophy*, *24*(3), 495-513;

[11] **Hanauske, Matthias, et al. (2010),** ***Doves and Hawks in Economics Revisited: An Evolutionary Quantum Game Theory Based Analysis of Financial Crises.*** *Physica A: Statistical Mechanics and its Applications, 389*(21), 5084-5102;

[12] **Brock, William A., Carsien Harm Hommes, and Florian Oskar Ottokar Wagener (2009),** ***More Hedging Instruments May Destabilize Markets.*** *Journal of Economic Dynamics and Control*, *33*(11), 1912-1928;

[13**] Hommes, C. H. (2006),** ***Heterogeneous Agent Models in Economics and Finance.*** *Handbook of computational economics, 2*, 1109-1186;

[14] **Smith, J. Maynard, and G. R. Price (1973),** ***The Logic of Animal Conflict.*** *Nature,246*, 15;

[15] **Jiang, Zhong-Ping, and Yuan Wang (2001),** ***Input-to-state stability for discrete-time nonlinear systems".*** *Automatica, 37*(6), 857-869.

[16] **Jiang, Zhong-Ping, and Yuan Wang. (2002*)." A converse Lyapunov theorem for discrete-time systems with disturbances".*** *Systems & control letters, 45*(1), 49-58;

[17] **Chang, Li-Chiu, Pin-An Chen, and Fi-John Chang (2012),** ***Reinforced Two-Step-Ahead Weight Adjustment Technique for Online Training of Recurrent Neural Networks.*** *Neural Networks and Learning Systems, IEEE Transactions on, 23*(8), 1269-1278;

[18] **Yu, Lean, Shouyang Wang, and Kin Keung Lai (2008*), An EMD-based Neural Network Ensemble Learning Model for World Crude Oil Spot Price Forecasting.*** *Soft Computing Applications in Business. Springer Berlin Heidelberg*, 261-271;

[19] **Zhou, Shifei, Kin Keung Lai, and Jerome Yen (2012),** ***A Dynamic Meta-Learning Rate-Based Model for Gold Market Forecasting.*** *Expert Systems with Applications, 39*(6), 6168-6173.